JOURNAL OF APPROXIMATION THEORY 34, 131–145 (1982)

Finite Dimensional Subspaces and Alternation

MANFRED SOMMER

Institut für Angewandte Mathematik der Universität Erlangen-Nürnberg, 8520 Erlangen, West Germany

Communicated by E. W. Cheney

Received December 18, 1980

It is shown that, for a class of finite dimensional subspaces G of $C(X)$, where X is a certain compact Hausdorff space, the following holds: For each $f \in C(X)$ there is a best uniform approximation $g_0 \in G$ for approximating f by G such that the error $f - g_0$ has at least $n+1$ extremal points ($\dim G = n$) which are oriented in a certain sense. Furthermore, in the case when X is any compact Hausdorff space, it is studied under which conditions on G, for each $f \in C(X)$, there exists at most one such best approximation and a sufficient condition for this is given.

INTRODUCTION

Let X denote a compact Hausdorff space and $C(X)$ the space of all real-valued continuous functions f on X under the uniform norm $\|f\| := \sup\{|f(x)| : x \in X\}$. If G is a finite dimensional subspace of $C(X)$, then, for each $f \in C(X)$, the set $P_G(f) := \{g_0 \in G : \|f - g_0\| = \inf\{\|f - g\| : g \in G\}\}$ is the *set of best uniform approximations* to f from G. It is well known that $P_G(f)$ is a singleton for each $f \in C(X)$ if and only if G satisfies the Haar condition. It is also well known (see Singer [7, p. 182]) that if $G = \text{span}\{g_1,..., g_n\}$ satisfies the Haar condition, then, for each $f \in C(X)$, the error $f - g_f$, where $\{g_f\} = P_G(f)$, has at least $n+1$ extremal points $x_0,..., x_n \in X$ such that $\varepsilon \cdot \varepsilon_i (-1)^i (f - g_f)(x_i) = \|f - g_f\|$, $i = 0,..., n$, $\varepsilon = \pm 1$, where

$$\varepsilon_i = \operatorname{sgn} \det(g_k(x_l))_{k=1}^{n}{}_{\substack{l=0 \\ l \neq i}}^{n}.$$

In the particular case when $X = [a, b]$, a real compact interval, and $x_0 < x_1 < \cdots < x_n \in [a, b]$, these constants assume the same values $\varepsilon_i = \tilde{\varepsilon}$ with $\tilde{\varepsilon} = \pm 1$ and, therefore, the error $f - g_f$ has at least $n+1$ *alternating* extreme points. Considering this special case we call g_f an *alternation element* of f also in case X is not an interval.

If G fails to satisfy the Haar condition, then for a given $f \in C(X)$, there may or may not exist a $g \in P_G(f)$ such that g can be interpreted as an alter-

0021-9045/82/020131–15\$02.00/0

nation element. This problem has been studied by some authors in the following sense: How to describe those spaces G for which, for each $f \in C(X)$, $X \subseteq \mathbb{R}$, there is a $g \in P_G(f)$ and $n+1$ distinct points $x_0 < x_1 < \cdots < x_n \in X$ such that $\varepsilon(-1)^i (f-g)(x_i) = \|f-g\|$, $i = 0,..., n$, $\varepsilon = \pm 1$?

In the particular case when $X = [a, b]$, Jones and Karlovitz [2] have completely solved this problem. They have shown that the spaces having this property are exactly the weak Chebyshev spaces. Later on, Deutsch *et al.* [1] have generalized this result to the weak Chebyshev subspaces of $C_0(T)$, where T is any locally compact subset of $\mathbb{R}$ and $C_0(T)$ denotes the Banach space of all real-valued continuous functions on T vanishing at infinity, i.e., for each $\varepsilon > 0$ the set $\{x \in T: |f(x)| \geqslant \varepsilon\}$ is compact. Nürnberger and Sommer [6] and Nürnberger [5] have characterized those weak Chebyshev subspaces of $C[a, b]$ and of $C_0(T)$, respectively, for which even uniqueness of the alternation elements holds.

In this paper we are concerned with finite dimensional subspaces of $C(X)$ which fail to satisfy the Haar condition and may *not* be weak Chebyshev. We study the problem of existence and uniqueness of alternation elements in this general case. Following the definition of alternation elements in the Haar case we analogously define such functions in the case when G belongs to a class of finite dimensional subspaces of $C(X)$ whose non-zero elements have at most finitely many zeros. We show (Theorem 1.5) that, under appropriate hypotheses on X, each $f \in C(X)$ has at least one alternation element for approximating f by G. We furthermore give a condition (Theorem 2.3) under which for each $f \in C(X)$ there is at most one alternation element. Then from both of these results there follows a result of Nürnberger and Sommer [6], their arguments, however, and also the arguments established by Nürnberger [5] do not apply to our case as we show in Example 1.

Our results immediately apply to the problem of existence of continuous selections for the metric projection P_G. Such a continuous selection s is a continuous mapping $s: C(X) \to G$ such that $s(f) \in P_G(f)$ for each $f \in C(X)$. We show in [8] that the property that each $f \in C(X)$ has a unique alternation element $g_f \in P_G(f)$, where X satisfies the same hypotheses as in Theorem 1.5 implies the existence of a continuous selection s defined by $s(f) := g_f$.

Using the arguments established in this paper it is easily verified that all results given here are also true if $C(X)$ will be replaced by $C_0(T)$, where T is a corresponding locally compact Hausdorff space.

1. Existence of Alternation Elements

In the following X will be any compact Hausdorff space and $\hat{X}$ a compact Hausdorff space satisfying the following property: For each sequence

$\{x_k\} \subset \hat{X}$ with $x_k \to x \in \hat{X}$ for $k \to \infty$ and each neighborhood U of x there is an integer k_1 such that for all $x_k, x_{\bar{k}} \in U$, $k \geqslant k_1$, $\bar{k} \geqslant k_1$, there is a path P from x_k to $x_{\bar{k}}$ completely contained in U.

Furthermore G will always denote an n-dimensional subspace of $C(X)$ and of $C(\hat{X})$, respectively, $n \geqslant 2$ and X, respectively, $\hat{X}$ will contain at least one non-isolated point. For brevity we will give some notations and definitions only for X but we will always assume that the same has been done for $\hat{X}$.

We often will use the following properties.

DEFINITION 1.1. We say that G satisfies the *Haar condition* on a subset Y of X if each non-zero $g \in G$ has at most $n-1$ zeros on Y. G is said to be *Chebyshev* if $P_G(f)$ is a singleton for each $f \in C(X)$.

It is well known that these both conditions are equivalent. In particular the following statement holds:

THEOREM 1.2. *The following statements are equivalent*:

(i) *G is Chebyshev.*

(ii) *G satisfies the Haar condition on X.*

(iii)

$$\det(g_i(x_j))_{i,j=1}^{n} := \begin{vmatrix} g_1(x_1) & \cdots & g_1(x_n) \\ \vdots & & \vdots \\ g_n(x_1) & \cdots & g_n(x_n) \end{vmatrix} \neq 0$$

for each basis $g_1, \ldots, g_n$ of G and all n distinct points $x_1, \ldots, x_n \in X$.

A proof of this classical result can be found in Meinardus [4].

Statement (iii) of the preceding theorem will play an important role for the following arguments. Therefore, for brevity we set

$$D_G(x_1, \ldots, x_n) := \det(g_i(x_j))_{i,j=1}^{n} \qquad \text{for all points } x_1, \ldots, x_n \in X,$$

where $g_1, \ldots, g_n$ is a fixed chosen basis of G.

Henceforth we will suppose that G satisfies the following conditions:

(1.1) There is a *minimal* finite subset $Z = \{z_1, \ldots, z_m\}$ of *non-isolated* points of X such that G satisfies the Haar condition on $X \setminus Z$.

(1.2) For any n distinct points $x_1, \ldots, x_n \in X$ there are pairwise disjoint neighborhoods U_i of x_i, $i = 1, \ldots, n$, such that $\varepsilon D_G(y_1, \ldots, y_n) \geqslant 0$, $\varepsilon = \pm 1$, for all $y_i \in U_i$, $i = 1, \ldots, n$.

Then these both conditions imply that for any n distinct points $x_1, \ldots, x_n \in X$ the inequality $\varepsilon D_G(y_1, \ldots, y_n) > 0$, $\varepsilon = \pm 1$, holds for all n-tuples $(y_1, \ldots, y_n) \in \prod_{i=1}^{n} U_i$ for which $\{y_1, \ldots, y_n\} \cap Z = \varnothing$.

If $X \subset \mathbb{R}$, then the finite dimensional subspaces G of $C(X)$ for which $\tilde{\varepsilon} D_G(x_1,...,x_n) \geqslant 0$, $\tilde{\varepsilon} = \pm 1$, for all points $x_1 < x_2 < \cdots < x_n \in X$ play a fundamental role in the approximation of functions. In particular, the following statements are equivalent:

(i) $\tilde{\varepsilon} D_G(x_1,..., x_n) \geqslant 0$, $\tilde{\varepsilon} = \pm 1$, for all points $x_1 < \cdots < x_n \in X$.

(ii) Each $g \in G$ has at most $n-1$ sign changes, i.e., there do not exist points $x_0 < x_1 < \cdots < x_n \in X$ such that $g(x_i)\, g(x_{i+1}) < 0$ for $i = 0,..., n-1$.

This equivalence has been proved by Jones and Karlovitz [2] if $X = [a, b]$, a real compact interval and by Deutsch *et al.* [1] if X is any compact subset of $\mathbb{R}$. By Karlin and Studden [3] a space G satisfying one of the preceding conditions is said to be a *weak Chebyshev* subspace of $C(X)$.

We will show that the conditions (1.1) and (1.2) imply that each $f \in C(X)$ has a particular best approximation. To do this we will need the following notations:

Let $x_0, x_1,..., x_n$ be any $n+1$ distinct points. If, for some $i \in \{0,..., n\}$, $Z \cap \{x_0,..., x_{i-1}, x_{i+1},..., x_n\} = \varnothing$, then we set:

$$\Delta_i(x_0,..., x_n) := \operatorname{sgn} D_G(x_0,..., x_{i-1}, x_{i+1},..., x_n).$$

If $Z \cap \{x_0,..., x_{i-1}, x_{i+1},..., x_n\} \neq \varnothing$, then by condition (1.2) there are neighborhoods U_j of x_j for $j = 0,..., n$, $j \neq i$, such that $\varepsilon D_G(y_0,..., y_{i-1}, y_{i+1},..., y_n) > 0$, $\varepsilon = \pm 1$, for all

$$(y_0,..., y_{i-1}, y_{i+1},..., y_n) \in \prod_{\substack{j=0 \\ j \neq i}}^{n} U_i$$

for which $\{y_0,..., y_{i-1}, y_{i+1},..., y_n\} \cap Z = \varnothing$. In this case we set

$$\Delta_i(x_0,..., x_n) := \operatorname{sgn} D_G(y_0,..., y_{i-1}, y_{i+1},..., y_n).$$

Then we define

DEFINITION 1.3. If $f \in C(X)$, then $g_0 \in P_G(f)$ is said to be an *alternation element* (AE) of f, if there exist $n+1$ distinct points $x_0,..., x_n \in X$ such that

$$\varepsilon(-1)^i \Delta_i(x_0,..., x_n)(f - g_0)(x_i) = \|f - g_0\|, \qquad i = 0,..., n, \quad \varepsilon = \pm 1.$$

The points $x_0,..., x_n$ are called *oriented extreme points* (*OE-points*) of $f - g_0$.

In the following the variables $x_0,..., x_n$ of Δ_i will sometimes be omitted.

Remark. If $X = [a, b]$ and G is an n-dimensional subspace of $C[a, b]$ satisfying the conditions (1.1) and (1.2), then it is easily verified that $\tilde{\varepsilon} D_G(x_1,..., x_n) \geqslant 0$, $\tilde{\varepsilon} = \pm 1$, for all points $a \leqslant x_1 < \cdots < x_n \leqslant b$. Then following the above given equivalence it turns out that G must be a weak Chebyshev subspace for which each nonzero $g \in G$ has at most finitely many zeros. Therefore we have that for all points $a \leqslant x_0 < x_1 < \cdots < x_n \leqslant b$ and $i = 0,..., n$ the relation $\Delta_i(x_0,..., x_n) = \tilde{\varepsilon}$ holds and in this case Definition 1.3 simplifies to the following: A function $g_0 \in P_G(f)$ is an AE of f, if there exist $n+1$ points $a \leqslant x_0 < \cdots < x_n \leqslant b$ such that $\varepsilon(-1)^i(f - g_0)(x_i) = \|f - g_0\|$, $i = 0,..., n$, $\varepsilon = \pm 1$.

In this case the points $x_0,..., x_n$ are alternating extreme points and therefore the notation AE seems to be justified also in our general situation. Jones and Karlovitz [2] have shown that the subspaces G of $C[a, b]$ for which for each $f \in C[a, b]$ there exists at least one AE $g_0 \in P_G(f)$ are *exactly* the weak Chebyshev subspaces of $C[a, b]$. Later on, Deutsch *et al.* [1] have generalized this result to the weak Chebyshev subspaces of $C_0(T)$, where T is any locally compact subset of the real line and a weak Chebyshev subspace G of $C_0(T)$ is defined analogously as in the case $X \subset \mathbb{R}$, X compact.

If G is a Chebyshev subspace of $C(X)$, then the existence of an AE for each $f \in C(X)$ is well known. This can be found in Singer [7, p. 182]. In particular, the following characterization of existence and uniqueness of best approximations is given:

THEOREM 1.4. *The following statements are equivalent*:

(i) *G is a Chebyshev space.*

(ii) *For every $f \in C(X)$ there exists a unique best approximation $g_f \in G$.*

(iii) *For every $f \in C(X)$ there exists a unique best approximation $g_f \in G$ and g_f is an AE of f.*

We are now able to prove our first result.

THEOREM 1.5. *Let G be an n-dimensional subspace of $C(\hat{X})$ satisfying the conditions* (1.1) *and* (1.2). *Then for each $f \in C(\hat{X})$ there exists at least one AE $g_0 \in P_G(f)$.*

Proof. Since each $z_l \in Z$ is non-isolated, for each $i = 1,..., m$ there is a sequence $\{z_{lk}\} \subset \hat{X}$ with $z_{lk} \to z_l$ for $k \to \infty$ and $z_{lk} \neq z_l$. This implies the existence of open neighborhoods U_{lk} of z_l, $i = 1,..., m$, such that for each k $U_{lk} \subset U_{l,k-1}$ and $z_{lk} \in U_{l,k-1} \backslash U_{lk}$. For each k we set $X_k := \hat{X} \backslash \bigcup_{l=1}^m U_{lk}$.

Then by condition (1.1), G satisfies the Haar condition on X_k and therefore Theorem 1.4 implies, for each $f \in C(\hat{X})$, the existence of a $g_k \in G$ such that $g_k|_{X_k}$ is an AE of $f|_{X_k}$ with respect to $G|_{X_k}$. Then it follows from $\|g_k\|_{X_k} \leqslant 2\|f\|$ that there is a subsequence of $\{g_k\}$ which we again denote by $\{g_k\}$ such that $g_k \to g_0$ for $k \to \infty$, $g_0 \in G$. This function g_0 satisfies the following:

(i) $g_0 \in P_G(f)$, since otherwise there is a $\bar{g} \in G$ with $\|f-\bar{g}\| < \|f-g_0\|$. But this is not possible, because $z_{ik} \to z_i$ and $z_{ik} \in X_k$ implies thé existence of an integer k_0 such that for all $k > k_0$ the inequality $\|f-\bar{g}\|_{X_k} < \|f-g_k\|_{X_k}$ holds.

(ii) g_0 is an AE of f. To show this, for each k we denote $n+1$ OE-points of $f-g_k$ by $x_{0k}, x_{1k}, ..., x_{nk}$. Then for each k the relation $\varepsilon_k(-1)^i \Delta_{ik}(f-g_k)(x_{ik}) = \|f-g_k\|_{X_k}$, $i=0,..., n$, $\varepsilon_k = \pm 1$, holds where $\Delta_{ik} := \operatorname{sgn} D_G(x_{0k},..., x_{i-1,k}, x_{i+1,k},..., x_{nk})$.

Without loss of generality we may assume that $\varepsilon_k = \varepsilon$ and $x_{ik} \to x_i \in \hat{X}$ for $k \to \infty$. If all points $x_0,..., x_n$ are distinct, then it follows from condition (1.2) that $\Delta_{ik} \to \Delta_i(x_0,..., x_n)$ for $k \to \infty$ and we are ready.

Therefore suppose that, for some $j \in \{0,..., n\}$ and some $l \in \{1,...,j\}$, $x_{j-l} = x_{j-l+1} = \cdots = x_j$ and $x_i \neq x_j$ for $i = 0,..., j-l-1, j+1,..., n$. We may assume that l is an odd number. Then it follows from $\varepsilon(-1)^{j-i} \Delta_{j-i,k}(f-g_k)(x_{j-i,k}) = \|f-g_k\|_{X_k} = \varepsilon(-1)^{j-i+1} \Delta_{j-i+1,k}(f-g_k)\,(x_{j-i+1,k})$ for $i = 1,..., l$ and $x_{j-i,k} \to x_{j-i} = x_j$ for $k \to \infty$, $i = 0,..., l$ that $\Delta_{j-l,k} = \Delta_{j-l+2,k} = \cdots = \Delta_{j-1,k} = -\Delta_{jk} = -\Delta_{j-2,k} = \cdots = -\Delta_{j-l+1,k}$ for k sufficiently large. Since $x_i \neq x_j$ for $i = 0,..., j-l-1, j+1,..., n$ there must be an integer k_0 and a neighborhood U of x_j such that $x_{j-l,k}, x_{j-l+1,k},..., x_{jk} \in U$ for all $k \geqslant k_0$ and $x_{ik} \notin U$ for $i = 0,..., j-l-1, j+1,..., n$. Then by definition of $\hat{X}$ there is an integer $k_1 \geqslant k_0$ such that for any two points $x_{\tilde{p}k}, x_{\tilde{q}k}, \tilde{p}, \tilde{q} \in \{j-1,..., j\}$, $k \geqslant k_1$, there is a path $\tilde{P}$ from $x_{\tilde{p}k}$ to $x_{\tilde{q}k}$ completely contained in U. Then it is easily verified that, for some $k \geqslant k_1$, there are two points x_{pk}, x_{qk} and a path $P \subset U$ from x_{pk} to x_{qk} such that $p, q \in \{j-l,..., j\}$, $p < q$, $p+q$ an odd number and $x_{ik} \notin P$ for $i = 0,..., n$, $i \neq p, q$. Since $\Delta_{pk} = -\Delta_{qk}$ and G satisfies the Haar condition on $\{x_{0k},..., x_{p-1,k}, x_{p+1,k},..., x_{nk}\}$ and on $\{x_{0k},..., x_{q-1,k}, x_{q+1,k},..., x_{nk}\}$ it follows that for the function l_k defined by

$$l_k(x) := D_G(x_{0k},..., x_{p-1,k}, x_{p+1,k},..., x_{q-1,k}, x_{q+1,k},..., x_{nk}, x)$$

the inequality

$$\begin{aligned} \operatorname{sgn} l_k(x_{pk}) \cdot \operatorname{sgn} l_k(x_{qk}) &= (-1)^{n-p-1} \Delta_{qk}(-1)^{n-q} \Delta_{pk} \\ &= \Delta_{qk} \cdot \Delta_{pk} = -1 < 0 \end{aligned}$$

must hold. This implies the existence of a point $\tilde{x} \in P$ such that $l_k(\tilde{x}) = 0$ and $\tilde{x}$ must be a zero with a sign change of l_k in $\hat{X}$. This means that for each neighborhood V of $\tilde{x}$ there are points $\bar{x}, \bar{\bar{x}} \in V$ such that

$$\begin{aligned} &D_G(x_{0k},..., x_{p-1,k}, x_{p+1,k},..., x_{q-1,k}, x_{q+1,k},..., x_{nk}, \bar{x}) \\ &\quad \times D_G(x_{0k},..., x_{p-1,k}, x_{p+1,k},..., x_{q-1,k}, x_{q+1,k},..., x_{nk}, \bar{\bar{x}}) < 0. \end{aligned}$$

Since by the preceding arguments all points $x_{0k},\dots, x_{p-1,k}, x_{p+1,k},\dots, x_{q-1,k}, x_{q+1,k},\dots, x_{nk}, \tilde{x}$ are distinct, we have got a contradiction to condition (1.2).

2. Uniqueness of Alternation Elements

We will show in [8] that the existence and the uniqueness of AEs guarantee the existence of continuous selections. Therefore in this section we study under which conditions on G for each $f \in C(\hat{X})$ there is a unique AE. In Theorem 2.2 we show that uniqueness holds if Z is a singleton. To prove this we need the following lemma which even holds for any compact Hausdorff space.

Lemma 2.1. *Let $Z = \{z\}$ and $g_0 \in P_G(f)$ be an* AE *of f. If $\{z\} \cap \{x_0,\dots, x_n\} \neq \varnothing$, where $x_0,\dots, x_n$ are $n+1$* OE*-points of $f - g_0$, then $g(z) = g_0(z)$ for all $g \in P_G(f)$.*

Proof. Let $x_0,\dots, x_n$ be $n+1$ OE-points of $f - g_0$. Then $\varepsilon(-1)^i \Delta_i(f - g_0)(x_i) = \|f - g_0\|$, $i = 0,\dots, n$, $\varepsilon = \pm 1$.

Let $g \in P_G(f)$ arbitrary. Then it follows from $\|f - g\| = \|f - g_0\|$ that $\varepsilon(-1)^i \Delta_i(f - g)(x_i) \leqslant \varepsilon(-1)^i \Delta_i(f - g_0)(x_i)$ and, therefore, $\varepsilon(-1)^i \Delta_i(g_0 - g)(x_i) \leqslant 0$, $i = 0,\dots, n$.

We now assume that there is a $j \in \{0,\dots, n\}$ with $z = x_j$. Then, since $G|_{X\setminus Z} = G|_{X\setminus\{z\}}$ satisfies the Haar condition, it follows that $D_G(x_0,\dots, x_{j-1}, x_{j+1},\dots, x_n) \neq 0$.

Since $\dim G = n$, we have the following equality:

$$0 = \begin{vmatrix} (g_0 - g)(x_0) & \cdots & (g_0 - g)(x_n) \\ g_1(x_0) & \cdots & g_1(x_n) \\ \vdots & & \vdots \\ g_n(x_0) & \cdots & g_n(x_n) \end{vmatrix}$$

$$= \sum_{i=0}^{n} (-1)^i D_G(x_0,\dots, x_{i-1}, x_{i+1},\dots, x_n)(g_0 - g)(x_i),$$

where $\{g_1,\dots, g_n\}$ is a fixed chosen basis of G and D_G is defined corresponding to this basis. Since for $D_G(x_0,\dots, x_{i-1}, x_{i+1},\dots, x_n) \neq 0$ the relation $\Delta_i = \operatorname{sgn} D_G(x_0,\dots, x_{i-1}, x_{i+1},\dots, x_n)$ holds, it follows from $\varepsilon(-1)^i \Delta_i(g_0 - g)(x_i) \leqslant 0$, $i = 0,\dots, n$, that for $i = 0,\dots, n$, $(-1)^i D_G(x_0,\dots, x_{i-1}, x_{i+1},\dots, x_n)(g_0 - g)(x_i) = 0$. Then $D_G(x_0,\dots, x_{j-1}, x_{j+1},\dots, x_n) \neq 0$ implies that $g(z) = g_0(z)$.

We are now in a position to give a sufficient condition for uniqueness of

AEs which shows that uniqueness of such particular best approximations is not restricted to the case when G is a Haar space.

THEOREM 2.2. *Let G be an n-dimensional subspace of $C(\hat{X})$ satisfying the conditions* (1.1) *and* (1.2) *with $Z=\{z\}$. Then each $f\in C(\hat{X})$ has a unique AE.*

The proof of this statement follows directly from Theorem 1.5 and the following theorem.

THEOREM 2.3. *Let G be an n-dimensional subspace of $C(X)$ satisfying the conditions* (1.1) *and* (1.2). *Then each $f\in C(X)$ has at most one AE.*

Proof. Suppose there is an $f\in C(X)$ having two AEs $g_0, g_1\in P_G(f)$. Let $x_0,\dots,x_n$ and $y_0,\dots,y_n$ be OE-points of $f-g_1$ and $f-g_0$, respectively. Then

$$\tilde{\varepsilon}(-1)^i \Delta_i(x_0,\dots,x_n)(f-g_1)(x_i)=\|f-g_1\|,\qquad i=0,\dots,n,\ \tilde{\varepsilon}=\pm 1,$$

and

$$\varepsilon(-1)^i \Delta_i(y_0,\dots,y_n)(f-g_0)(y_i)=\|f-g_0\|,\qquad i=0,\dots,n,\ \varepsilon=\pm 1.$$

Without loss of generality we may assume that $g_1\equiv 0$ and $\tilde{\varepsilon}=1$. We distinguish two cases.

First case. $z\notin\{x_0,\dots,x_n\}$ or $z\notin\{y_0,\dots,y_n\}$. Without loss of generality let $z\notin\{x_0,\dots,x_n\}$. Then condition (1.1) implies that G satisfies the Haar condition on $\{x_0,\dots,x_n\}$. Furthermore by the arguments of Lemma 2.1 we have that 0 is an AE of f for approximation by G on $\{x_0,\dots,x_n\}$ and $|(f-g_0)(x_i)|\leqslant|f(x_i)|$ for $i=0,\dots,n$. But this contradicts the statements of Theorem 1.4.

Therefore we have only to consider the second case.

Second case. $z\in\{x_0,\dots,x_n\}$ and $z\in\{y_0,\dots,y_n\}$. Let $z=x_j=y_k$. Then Lemma 2.1 implies that $g(x_j)=0$ for all $g\in P_G(f)$ and, in particular, $f(x_j)=(f-g_0)(y_k)$.

In the following we will only need a special subset $\tilde{X}$ of X. We set

$$\tilde{X}:=\{x_0,\dots,x_n,y_0,\dots,y_n\}\cup\bar{U}_0,$$

where $\bar{U}_0$ is a closed neighborhood of z in X such that

$$\{x_0,\dots,x_{j-1},x_{j+1},\dots,x_n,y_0,\dots,y_{k-1},y_{k+1},\dots,y_n\}\cap\bar{U}_0=\varnothing$$

and $|g_0(x)|<\frac{1}{2}|f(x)|$ for all $x\in\bar{U}_0$ (remember that $g_0(z)=0$). We will now construct a function $\tilde{f}\in C(\tilde{X})$ and an n-dimensional subspace $\tilde{G}$ of $C(\tilde{X})$ such that the following conditions hold:

(i) $\tilde{G}$ satisfies the Haar condition on $\tilde{X}\backslash\{z\}$.

(ii) $\operatorname{sgn} D_{\tilde{G}}(t_1,..., t_n) = \operatorname{sgn} D_G(t_1,..., t_n)$ for all points $t_1,..., t_n \in \tilde{X}$ and certain bases $g_1,..., g_n$ of G and $\tilde{g}_1,..., \tilde{g}_n$ of $\tilde{G}$.

(iii) $\tilde{f}$ has two AEs $0, \tilde{g}_0 \in \tilde{G}$. Furthermore $x_0,..., x_n$ are $n+1$ OE-points of $\tilde{f}-0$ and $y_0,..., y_n$ are $n+1$ OE-points of $\tilde{f}-\tilde{g}_0$.

(iv) If $y_i \notin \{x_0,..., x_n\}$, then $|\tilde{f}(y_i)| < \|\tilde{f}\|_{\tilde{X}}$ and if $x_i \notin \{y_0,..., y_n\}$, then $|(\tilde{f}-\tilde{g}_0)(x_i)| < \|\tilde{f}-\tilde{g}_0\|_{\tilde{X}}$.

Before constructing such a function $\tilde{f}$ and a subspace $\tilde{G}$ having the preceding properties we show that the existence of $\tilde{f}$ and $\tilde{G}$ yields a contradiction of the hypothesis that f has two AEs $0, g_0 \in P_G(f)$. Since by condition (1.1) the point z is non-isolated, there is a sequence $\{z_m\} \subset \bar{U}_0$ such that $z_m \to z$ for $m \to \infty$, $z_m \neq z$. This implies that $\{z_m\} \cap \{x_0,..., x_n, y_0,..., y_n\} = \varnothing$. For each $m \in \mathbb{N}$ we set:

$$T_{1m} := \{x_0,..., x_{j-1}, z_m, x_{j+1},..., x_n\}$$

and

$$T_{2m} := \{y_0,..., y_{k-1}, z_m, y_{k+1},..., y_n\}.$$

We now approximate $\tilde{f}$ by $\tilde{G}$ on T_{1m} and T_{2m}. Since $\tilde{G}$ satisfies the Haar condition on T_{1m} and on T_{2m}, following Theorem 1.4, there must exist a unique best approximation $h_m \in \tilde{G}$ for approximating $\tilde{f}$ on T_{1m} and a unique best approximation $g_m \in \tilde{G}$ for approximating $\tilde{f}$ on T_{2m}. Furthermore h_m is an AE of $\tilde{f}$ on T_{1m} and g_m is an AE of $\tilde{f}$ on T_{2m}. Since $z_m \to z$ for $m \to \infty$ it is easily verified that $h_m \to 0$ and $g_m \to \tilde{g}_0$ for $m \to \infty$.

Now let $m_0 \in \mathbb{N}$ be sufficiently large such that $h_m \neq g_m$ for all $m \geqslant m_0$. We distinguish:

(i) $\|\tilde{f}-h_m\|_{T_{1m}} \leqslant \|\tilde{f}-g_m\|_{T_{2m}}$. If $y_i \notin \{x_0,..., x_n\}$, then $y_i \neq z$ and it follows from the construction of $\tilde{f}$ that $|\tilde{f}(y_i)| < |(\tilde{f}-\tilde{g}_0)(y_i)| = \|\tilde{f}-\tilde{g}_0\|_{\tilde{X}}$. This implies the existence of an integer m_1 such that, for all $m \geqslant m_1$, $|(\tilde{f}-h_m)(y_i)| < |(\tilde{f}-g_m)(y_i)|$. If $y_i \in \{x_0,..., x_n\}$ and $y_i \neq z$, then $|(\tilde{f}-h_m)(y_i)| = \|\tilde{f}-h_m\|_{T_{1m}} \leqslant \|\tilde{f}-g_m\|_{T_{2m}} = |(\tilde{f}-g_m)(y_i)|$.

Furthermore $|(\tilde{f}-h_m)(z_m)| = \|\tilde{f}-h_m\|_{T_{1m}} \leqslant \|\tilde{f}-g_m\|_{T_{2m}} = |(\tilde{f}-g_m)(z_m)|$. Thus we have shown that for all $m \geqslant m_1$ $\|\tilde{f}-h_m\|_{T_{2m}} \leqslant \|\tilde{f}-g_m\|_{T_{2m}}$. Then h_m is also a best approximation for $\tilde{f}$ on T_{2m} which contradicts the hypothesis that $\tilde{G}$ satisfies the Haar condition on T_{2m}.

(ii) $\|\tilde{f}-h_m\|_{T_{1m}} > \|\tilde{f}-g_m\|_{T_{2m}}$. Here we can conclude as in case (i).

Therefore we still must show the existence of a function $\tilde{f} \in C(\tilde{X})$ and of an n-dimensional subspace $\tilde{G}$ of $C(\tilde{X})$ satisfying conditions (i) to (iv). If conditions (iii) and (iv) already hold for the functions $f-0$ and $f-g_0$, then we set $\tilde{G} := G$, $\tilde{f} := f$, $\tilde{g}_0 := g_0$ and the proof is complete.

If not, then we define the following subsets of $\{x_0,\dots,x_n\}$, respectively $\{y_0,\dots,y_n\}$:

$$\begin{aligned}
E_1 &:= \{x_i : (f-g_0)(x_i) = -f(x_i), x_i \notin \{y_0,\dots,y_n\}\},\\
F_1 &:= \{y_i : f(y_i) = -(f-g_0)(y_i), y_i \notin \{x_0,\dots,x_n\}\},\\
E_2 &:= \{x_i : (f-g_0)(x_i) = -f(x_i), x_i \in \{y_0,\dots,y_n\}\}\\
&= \{y_i : (f-g_0)(y_i) = -f(y_i), y_i \in \{x_0,\dots,x_n\}\},\\
E_3 &:= \{x_i : (f-g_0)(x_i) = f(x_i), x_i \notin \{y_0,\dots,y_n\}\},\\
F_3 &:= \{y_i : (f-g_0)(y_i) = f(y_i), y_i \notin \{x_0,\dots,x_n\}\},\\
E_4 &:= \{x_i : (f-g_0)(x_i) = f(x_i), x_i \in \{y_0,\dots,y_n\}\},\\
E_5 &:= \{x_i : |(f-g_0)(x_i)| < |f(x_i)| = \|f\|\},\\
F_5 &:= \{y_i : |f(y_i)| < |(f-g_0)(y_i)| = \|f-g_0\|\}.
\end{aligned}$$

Then $x \in E_1 \cup F_1 \cup E_2$ implies that $g_0(x) = 2f(x)$ and $x \in E_3 \cup F_3 \cup E_4$ implies that $g_0(x) = 0$. Furthermore it follows from $z = x_j = y_k$ that $z \in E_4$. We may assume that $E_1 = \varnothing$ and $F_1 = \varnothing$. Otherwise we define a function $\bar{f} \in C(\tilde{X})$ by

$$\begin{aligned}
\bar{f}(x_i) &:= f(x_i) && \text{for all } x_i \in E_1,\\
\bar{f}(y_i) &:= 0 && \text{for all } y_i \in F_1,\\
\bar{f}(x) &:= f(x) && \text{for all } x \in \tilde{X} \setminus (E_1 \cup F_1).
\end{aligned}$$

We furthermore define a subspace $\bar{G}$ of $C(\tilde{X})$ by

$$\begin{aligned}
\bar{G} := \{\bar{g} \in C(\tilde{X})\colon &\text{There exists a } g \in C|_{\tilde{X}} \text{ such that}\\
\bar{g}(x) &= \tfrac{1}{2} g(x) \quad \text{if } x \in E_1 \cup F_1\\
&= g(x) \quad \text{otherwise}\}.
\end{aligned}$$

Then the following properties are easily verified:

(i) $\bar{G}$ is an n-dimensional subspace of $C(\tilde{X})$ and satisfies the Haar condition on $\tilde{X} \setminus \{z\}$.

(ii) $\operatorname{sgn} D_{\bar{G}}(t_1,\dots,t_n) = \operatorname{sgn} D_G(t_1,\dots,t_n)$ for all points $t_1,\dots,t_n \in \tilde{X}$, where D_G is defined corresponding to a fixed chosen basis $g_1,\dots,g_n$ of G and $D_{\bar{G}}$ corresponding to the basis $\bar{g}_1,\dots,\bar{g}_n$ with

$$\begin{aligned}
\bar{g}_i(x) &= \tfrac{1}{2} g_i(x) \quad \text{if } x \in E_1 \cup F_1\\
&= g_i(x) \quad \text{otherwise}
\end{aligned}, \qquad i = 1,\dots,n.$$

(iii) Using the arguments in the proof of Lemma 2.1 it follows that $\bar{f}$ has the two AEs $0, \bar{g}_0 \in \bar{G}$ for approximation in $\tilde{X}$ where by definition of $\bar{G}$,

$$\begin{aligned}
\bar{g}_0(x) &= \tfrac{1}{2} g_0(x) \quad \text{if } x \in E_1 \cup F_1\\
&= g_0(x) \quad \text{otherwise.}
\end{aligned}$$

Furthermore $x_0,\dots,x_n$ are $n+1$ OE-points of $\bar{f}-0$ and $y_0,\dots,y_n$ are $n+1$ OE-points of $\bar{f}-\bar{g}_0$.

If we denote the corresponding subsets of $\{x_0,\dots,x_n\}$ and of $\{y_0,\dots,y_n\}$ to the functions $\bar{f}-0$ and $\bar{f}-\bar{g}_0$ by $\bar{E}_i, \bar{F}_i$ instead of E_i, F_i, then it immediately follows from the construction of $\bar{f}, \bar{g}_0$ that $\bar{E}_1=\bar{F}_1=\varnothing$, $\bar{E}_i=E_i$ for $i=2, 3, 4$, $\bar{F}_3=F_3$, $\bar{E}_5=E_5\cup E_1$, $\bar{F}_5=F_5\cup F_1$. Therefore we may assume that $E_1=\varnothing$, $F_1=\varnothing$.

We may now complete the proof by constructing a function $\tilde{f}$ and a subspace $\tilde{G}$ having the desired properties. We first set $\{t_0,\dots,t_r\} := E_3\cup F_3\cup E_4$ and observe that $r+1\leqslant n$, because by definition of E_3, F_3, E_4 for each $i=0,\dots,r$, t_i must be a zero of g_0, but each non-zero $g\in G$ has at most n distinct zeros. Since $z\in E_4$ we assume that $z=t_0$. We now choose $n-r$ distinct points $t_{r+1},\dots,t_n\in E_2\cup E_5$. Then $z=t_0\notin\{t_1,\dots,t_n\}$ implies that $D_G(t_1,\dots,t_n)\neq 0$ and, therefore, for each $m\in\mathbb{N}$ there is a $g_{0m}\in G$ such that

$$\begin{aligned} g_{0m}(t_i) &:= g_0(t_i) && \text{if } t_i\in E_2\cup E_4\cup E_5,\ t_i\neq z,\\ &= \frac{(-1)^l \Delta_l(x_0,\dots,x_n)}{m} && \text{if } t_i=x_l\in E_3,\\ &= \frac{-\varepsilon(-1)^l \Delta_l(y_0,\dots,y_n)}{m} && \text{if } t_i=y_l\in F_3. \end{aligned}$$

Then it is easily verified that $g_{0m}\to g_0$ for $m\to\infty$. Therefore there is an integer m_0 such that for all $x\in E_5$ $|(f-g_{0,m_0})(x)|<|f(x)|$ and for all $x\in F_5$ $|f(x)|<|(f-g_0)(x)|-|(g_0-g_{0,m_0})(x)|$. Furthermore let m_0 be so sufficiently large that for all $x\in\bar{U}_0$ $|g_{0,m_0}(x)|<|f(x)|$ and for all $x\in E_2$ $\operatorname{sgn} g_0(x)=\operatorname{sgn} g_{0,m_0}(x)$ and $1/m_0<\|f\|$. We set $\hat{g}_0 := g_{0,m_0}$.

Furthermore we define a function $\tilde{f}$ by

$$\tilde{f}(x) := f(x) \qquad \text{for all } x\in E_2\cup E_3\cup E_4\cup E_5,\quad x\neq z$$

$$\tilde{f}(y_i) := f(y_i)-\frac{\varepsilon(-1)^i \Delta_i(y_0,\dots,y_n)}{m_0} \qquad \text{for all } y_i\in F_3$$

$$\tilde{f}(x) := f(x)-g_0(x)+\hat{g}_0(x) \qquad \text{for all } x\in F_5.$$

On the set $\bar{U}_0$, $\tilde{f}$ will be defined later.

Since for all $x_l\in E_2$, $\operatorname{sgn}\hat{g}_0(x_l)=\operatorname{sgn} g_0(x_l)$, for all $x_l\in E_2$ there are positive real numbers c_l such that $c_l\,\hat{g}_0(x_l)=g_0(x_l)$. Using these numbers we define a subspace $\tilde{G}$ of $C(\tilde{X})$ by

$$\begin{aligned} \tilde{G} := \{\tilde{g}\in C(\tilde{X})\colon &\text{there is a } g\in G \text{ such that}\\ \tilde{g}(x) &= g(x) && \text{if } x\in\tilde{X}\setminus E_2\\ &= c_l\, g(x) && \text{if } x=x_l\in E_2\}. \end{aligned}$$

Then it is easily verified that $\tilde{G}$ is an n-dimensional space satisfying the Haar condition on $\tilde{X}\backslash\{z\}$. Furthermore it follows from $c_i > 0$ that $\operatorname{sgn} D_{\tilde{G}}(t_1,..., t_n) = \operatorname{sgn} D_G(t_1,..., t_n)$ for all points $t_1,..., t_n \in \tilde{X}$. This implies that $\tilde{\Delta}_i(x_0,..., x_n) = \Delta_i(x_0,..., x_n)$ for $i = 0,..., n$, where $\tilde{\Delta}_i$ is defined with respect to $\tilde{G}$ analogously as Δ_i.

Now considering the functions $\tilde{f} - 0$ and $\tilde{f} - \tilde{g}_0$ on $\{x_0,..., x_n, y_0,..., y_n\}\backslash\{z\}$, where $\tilde{g}_0 \in \tilde{G}$ belongs to $\hat{g}_0$ with respect to $\tilde{G}$, i.e.,

$$\begin{aligned} \tilde{g}_0(x) &= \hat{g}_0(x) && \text{if} \quad x \in \tilde{X}\backslash E_2 \\ &= c_i \hat{g}_0(x) && \text{if} \quad x = x_i \in E_2 \end{aligned}$$

it is easy to show that

$$\begin{aligned} (-1)^i \tilde{\Delta}_i(x_0,..., x_n)\tilde{f}(x_i) &= \|f\|, && i = 0,..., n, \quad i \neq j, \\ \varepsilon(-1)^i \tilde{\Delta}_i(y_0,..., y_n)(\tilde{f} - \tilde{g}_0)(y_i) &= \|f - g_0\|, && i = 0,..., n, \quad i \neq k. \end{aligned}$$

Therefore we still have to define $\tilde{f}$ on $\bar{U}_0$ such that z is also an OE-point of $\tilde{f} - 0$ and of $\tilde{f} - \tilde{g}_0$. Without loss of generality let $f(z) = \|f\|$. We distinguish:

(i) $\tilde{g}_0(z) \geqslant 0$. Since $\tilde{g}_0(z) = \hat{g}_0(z) = g_{0,m_0}(z)$ and $m_0 \in \mathbb{N}$ has been chosen such that $|g_{0,m_0}(x)| < |f(x)|$ for all $x \in \bar{U}_0$, it follows that $0 \leqslant \tilde{g}_0(z) < f(z)$.

We set $\tilde{f}(z) := f(z)$ and define $\tilde{f}$ on $\bar{U}_0$ such that, for all $x \in \bar{U}_0$, $|\tilde{f}(x)| \leqslant |\tilde{f}(z)|$ and $|(\tilde{f} - \tilde{g}_0)(x)| \leqslant |\tilde{f}(z)|$ and $\tilde{f} \in C(\tilde{X})$.

This implies that $(-1)^i \tilde{\Delta}_i(x_0,..., x_n)\tilde{f}(x_i) = \|\tilde{f}\|_{\tilde{X}}$ for $i = 0..., n$. Then following the proof of Lemma 2.1 we can easily show that $0 \in P_{\tilde{G}}(\tilde{f})$. Furthermore by the preceding arguments we have that $\tilde{g}_0 \in P_{\tilde{G}}(\tilde{f})$, too. Then it follows from Lemma 2.1 that $\tilde{g}_0(z) = 0$ and therefore

$$\varepsilon(-1)^i \tilde{\Delta}_i(y_0,..., y_n)(\tilde{f} - \tilde{g}_0)(y_i) = \|\tilde{f} - \tilde{g}_0\|_{\tilde{X}} = \|\tilde{f}\|_{\tilde{X}} = \|f\|.$$

(ii) $\tilde{g}_0(z) < 0$. We set $\tilde{f}(z) := f(z) + \tilde{g}_0(z)$ and define $\tilde{f}$ on $\bar{U}_0$ such that for all $x \in \bar{U}_0$, $|(\tilde{f} - \tilde{g}_0)(x)| \leqslant |(\tilde{f} - \tilde{g}_0)(z)| = |f(z)| = \|f\|$ and $|\tilde{f}(x)| \leqslant |(\tilde{f} - \tilde{g}_0)(z)|$ and $\tilde{f} \in C(\tilde{X})$. Exactly as in case (i) we can show that

$$\varepsilon(-1)^i \tilde{\Delta}_i(y_0,..., y_n)(\tilde{f} - \tilde{g}_0)(y_i) = \|\tilde{f} - \tilde{g}_0\|_{\tilde{X}} = \|\tilde{f}\|_{\tilde{X}}, \qquad i = 0,..., n.$$

We have only to consider that because of $g_0(z) = 0$ $\operatorname{sgn}(\tilde{f} - \tilde{g}_0)(z) = \operatorname{sgn} f(z) = \operatorname{sgn}(f - g_0)(z)$. Then we can show again that $0, \tilde{g}_0 \in P_{\tilde{G}}(\tilde{f})$. But this contradicts the statement of Lemma 2.1 because for all $g \in P_{\tilde{G}}(\tilde{f})$ the relation $g(z) = \tilde{g}_0(z) < 0$ must be valid. Therefore $\tilde{g}_0(z) < 0$ is not possible.

Thus we have shown that $\tilde{g}_0(z) = 0$ and we have defined an $\tilde{f} \in C(\tilde{X})$ having two AEs $0, \tilde{g}_0 \in \tilde{G}$. Furthermore it is readily verified that $\tilde{E}_1 = \tilde{F}_1 = \tilde{E}_3 = \tilde{F}_3 = \varnothing$, where these subsets of $\{x_0,..., x_n\}$ and of $\{y_0,..., y_n\}$ are defined to $\tilde{f}-0$ and $\tilde{f}-\tilde{g}_0$ analogously as the sets E_i, F_i to the functions $f-0$ and $f-g_0$. But this is equivalent to the following:

$$\text{If} \quad x_i \notin \{y_0, ..., y_n\}, \qquad \text{then} \quad |(\tilde{f}-\tilde{g}_0)(x_i)| < |\tilde{f}(x_i)|$$

and

$$\text{if} \quad y_i \notin \{x_0, ..., x_n\}, \qquad \text{then} \quad |\tilde{f}(y_i)| < |(\tilde{f}-\tilde{g}_0)(y_i)|.$$

As has been shown above the existence of such functions $\tilde{f}$, 0, $\tilde{g}_0$ is impossible. This completes the proof.

In the particular case $X=[a, b]$, the statement of Theorem 2.2 has been proved by Nürnberger and Sommer [6] (see also Sommer and Strauss [9]). If T is any locally compact subset of $\mathbb{R}$ and $G \subset C_0(T)$ is weak Chebyshev (here weak Chebyshev is defined analogously as in the case $X=[a, b]$) then the statement of Theorem 2.2 follows directly from a result of Nürnberger [5]. For proving his result this author has observed that the problem must only be studied on certain sets of alternating extreme points of error functions $f-g_0$ and $f-\tilde{g}_0$, where $g_0, \tilde{g}_0 \in G$ are assumed to be AEs of a function $f \in C_0(T)$. Therefore for $X \subset \mathbb{R}$ the arguments established in that paper would apply to our case if we can transform the given subspace G into a weak Chebyshev subspace on the sets of OE-points of certain error functions $f-g_0$ and $f-\tilde{g}_0$ by changing the sign of the basis functions of G on these sets. Unfortunately this is not true in general as the following example shows.

EXAMPLE 1. Let $X=[0, 1] \cup [2, 3]$ and the two functions $g_1, g_2 \in C(X)$ be defined by $g_1(x) := 1$ and

$$\begin{aligned} g_2(x) &:= x \qquad && \text{if} \quad x \in [0, 1] \\ &= 2-x \qquad && \text{if} \quad x \in [2, 3]. \end{aligned}$$

Then the space $G := \text{span}\{g_1, g_2\}$ satisfies the Haar condition on $X \setminus \{0\}$ and condition (1.2), too. But G is not weak Chebyshev, since the function $g_2 - \frac{1}{2} g_1$ has two sign changes. However, Theorem 2.2 implies that each $f \in C(X)$ has exactly one AE. If we try to prove the statement of this theorem by following the arguments in [5], we would suppose that there is an $f \in C(X)$ having two AEs $g_0, \tilde{g}_0 \in G$ with OE-points x_0, x_1, x_2 and y_0, y_1, y_2, respectively. For example, the partition $x_0=0$, $x_1=1$, $x_2=2$, $y_0=0$, $y_1=2$, $y_2=3$ could be possible. But the arguments in [5] only apply to our case if G can be transformed into a weak Chebyshev subspace on the set $\{x_0, x_1, x_2, y_0, y_1, y_2\} = \{0, 1, 2, 3\}$ by changing the sign of g_1 and g_2 on this set. However, it is easily verified that there do not exist any numbers σ_i, $\tau \in \{-1, 1\}$, $i=0, 1, 2$, such that the space $\tilde{G}$ defined by $\tilde{G} := \text{span}\{\tilde{g}_1, \tilde{g}_2\}$, where for $j = 1, 2$, $\tilde{g}_j(x_i) := \sigma_i g_j(x_i)$, $i = 0, 1, 2$, and $\tilde{g}_j(y_2) := \tau g_j(y_2)$ is weak Chebyshev on $\{0, 1, 2, 3\}$.

In the case $X = [a, b]$ the results in [6] and [9] show that the converse to Theorem 2.2 is also true. This is a consequence of the following result established by Sommer and Strauss [9]:

THEOREM 2.4. *The following statements are equivalent:*

(i) *G is a weak Chebyshev subspace of $C[a, b]$ and each non-zero $g \in G$ has at most n distinct zeros.*

(ii) *G is weak Chebyshev and there is an $\tilde{x} \in [a, b]$ such that G satisfies the Haar condition on $[a, b] \setminus \{\tilde{x}\}$.*

If we replace weak Chebyshev by condition (1.2) in our general situation then a corresponding statement is unfortunately no longer true as the following example shows.

EXAMPLE 2. Let $X = [0, 1] \cup [2, 3] \cup [4, 5]$ and the two functions $g_1, g_2 \in C(X)$ be defined by

$$g_1(x) := \begin{cases} 1 & \text{if } x \in [0, 1] \\ -1 & \text{if } x \in [2, 3] \\ x - 5 & \text{if } x \in [4, 5] \end{cases} \quad \text{and} \quad g_2(x) := \begin{cases} x & \text{if } x \in [0, 1] \\ x - 2 & \text{if } x \in [2, 3] \\ -1 & \text{if } x \in [4, 5]. \end{cases}$$

Let $G := \text{span}\{g_1, g_2\}$. Then each $g \in G$ has at most two distinct zeros and at most one zero with a sign change in X. Therefore by Lemma 2.2 in [8] G satisfies conditions (1.1) and (1.2). However, observing that g_2 has the zeros $x_1 = 0$, $x_2 = 2$ and $g_1 - g_2$ has the zeros $x_1 = 1$, $x_2 = 4$, we have that there is no point $z \in X$ such that G satisfies the Haar condition on $X \setminus \{z\}$. Looking for a minimal set Z guaranteeing condition (1.1) we can choose $Z = \{0, 1\}$, $Z = \{0, 4\}$, $Z = \{1, 2\}$ or $Z = \{2, 4\}$.

Therefore we conjecture that the statement of Theorem 2.2 holds for a greater class of subspaces.

Conjecture. Let G satisfy conditions (1.1) and (1.2) and let each non-zero $g \in G$ have at most n distinct zeros. Then for each $f \in C(X)$ there exists a unique AE.

The hypothesis that each non-zero $g \in G$ has at most n distinct zeros cannot be weakened. This is easily verified by using the arguments established in the proof of Theorem 11 in [6] and we get the following converse to the preceding conjecture.

THEOREM 2.5. *Let G satisfy conditions* (1.1) *and* (1.2) *and let each $f \in C(X)$ have a unique AE. Then each non-zero $g \in G$ has at most n distinct zeros.*

REFERENCES

1. F. DEUTSCH, G. NÜRNBERGER AND I. SINGER, Weak Chebyshev subspaces and alternation, *Pacific J. Math.* **89** (1980), 9–31.
2. R. C. JONES AND L. A. KARLOVITZ, Equioscillation under nonuniqueness in the approximation of continuous functions, *J. Approx. Theory* **3**(1970), 138–145.
3. S. KARLIN AND W. J. STUDDEN, "Tchebycheff Systems: With Applications in Analysis and Statistics," Interscience, New York, 1966.
4. G. MEINARDUS, "Approximation of Functions: Theory and Numerical Methods," Springer-Verlag, Berlin, 1967.
5. G. NÜRNBERGER, Continuous Selections for the metric projection and alternation, *J. Approx. Theory* **28** (1980), 212–226.
6. G. NÜRNBERGER AND M. SOMMER, Weak Chebyshev subspaces and continuous selections for the metric projection, *Trans. Amer. Math. Soc.* **238** (1978), 129–138.
7. I. SINGER, "Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces," Springer-Verlag, Berlin, 1970.
8. M. SOMMER, Existence of pointwise-Lipschitz-continuous selections of the metric projection for a class of Z-spaces, *J. Approx. Theory* **34** (1982), 115–130.
9. M. SOMMER AND H. STRAUSS, Eigenschaften von schwach tschebyscheffschen Räumen, *J. Approx. Theory* **21**(1977), 257–268.